Lecture 13

Perturbation Theory

Recap

Where were we?

General Relativity

Actions

Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}} \longleftarrow \boxed{S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_M}$$

Geodesic equations

$$\boxed{\frac{dx^\nu}{d\tau} \nabla_\nu \frac{dx^\mu}{d\tau} = 0} \longleftarrow \boxed{S = -m \int d\tau}$$

Energy-momentum tensor

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} - \frac{\Lambda}{8\pi G} g_{\mu\nu}}$$

Cosmological constant

$$\frac{\Lambda}{8\pi G} = \rho_{vac}$$

Plan for today

- Linearized Einstein equations
- Light deflection
- Gravitational waves
- Laws of Black Hole Mechanics (cont. of previous lecture)

Linearized Gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad , \qquad |h_{\mu\nu}| << 1$$

We shall neglect $O(h^2)$ in the e.o.m.

Exercise: check that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad , \qquad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho \Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho}$$

$$= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\right)$$

Gauge transformations = coordinate transformations

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}$$

Exercise: Show that the infinitesimal coord. transf.

$$x^{\mu} \to x^{\mu} - \xi^{\mu}(x)$$

leads to (in the linearized theory)

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}$$

Exercise: Check that the linearized Riemann Tensor is invariant

$$R_{\mu\nu\rho\sigma} \to R_{\mu\nu\rho\sigma}$$

[Analogous to $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ being inv. under $A_{\mu} \to A_{\mu} + \partial_{\mu} \chi$]

Linearized Einstein Equations

It is convenient to split the metric perturbation $h_{\mu\nu}$ into irreducible tensors under spatial rotations $SO(3)$:

$$h_{00} = \underset{\uparrow \atop \text{scalar}}{-2\Phi} \,, \quad h_{0i} = \underset{\uparrow \atop \text{vector}}{w_i} \,, \quad h_{ij} = 2\underset{\uparrow \atop \text{traceless sym. tensor (spin 2)}}{S_{ij}} - 2\underset{\uparrow \atop \text{scalar}}{\Psi}\delta_{ij}$$

$$\rightarrow \quad \boxed{ds^2 = -(1+2\Phi)\,dt^2 + 2\,w_i\,dx^i\,dt + \left[(1-2\Psi)\delta_{ij} + 2S_{ij}\right]dx^i dx^j}$$

Under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, we have

$$S_{ij} \to S_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\delta_{ij}\partial_k\xi^k \qquad\qquad w_i \to w_i + \partial_0\xi^i - \partial_i\xi^0$$

∂^i

$$\partial^i S_{ij} \to \partial^i S_{ij} + \frac{1}{2}\underset{\partial_k\partial^k}{\underset{\parallel}{\nabla^2}}\xi_j + \frac{1}{6}\partial_j\partial^i\xi_i \underset{\uparrow \atop \text{choice of } \xi_j}{=} 0 \qquad\qquad \partial^i w_i \to \partial^i w_i + \partial_0\partial_i\xi^i - \nabla^2\xi^0 \underset{\uparrow \atop \text{choice of } \xi_0}{=} 0$$

Transverse gauge

Linearized Einstein Equations

In transverse gauge $[\partial_i w^i = \partial_i s^{ij} = 0]$, E.E. read: [Check!]

$$\boxed{\begin{aligned} G_{00} &= 2\nabla^2\Psi = 8\pi G T_{00} \\ G_{0j} &= -\tfrac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j} \\ G_{ij} &= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij} \end{aligned}}$$

from $\delta^{ij} G_{ij} = 8\pi G\, \delta^{ij} T_{ij}$

Degrees of freedom: Ψ, w_j, Φ are not propagating d.o.f. because they obey eq. of the type $\nabla^2\Psi = $ (something ind. of Ψ).

On the other hand,

$\Box s_{ij} = (\dots)$

wave op.

$s_{ij} \to$ sym. traceless $\to$ $6 - 1 = 5$

$\partial^i s_{ij} = 0 \to 3$ eqs. $\Rightarrow$ $5 - 3 = 2$ d.o.f.

Newtonian fields

Consider static sources. More precisely, we consider dust ($p \ll \rho$)

$$T_{\mu\nu} = \rho U_\mu U_\nu \quad , \quad U^\mu = (1,0,0,0) \quad \text{rest frame}$$

$$\boxed{\begin{aligned} G_{00} &= 2\nabla^2\Psi = 8\pi G T_{00} \\ G_{0j} &= -\tfrac{1}{2}\nabla^2 w_j + \cancel{2\partial_0\partial_j\Psi} = \cancel{8\pi G T_{0j}} \\ * \quad G_{ij} &= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\Psi) - \cancel{\partial_0\partial_{(i}w_{j)}} + \cancel{2\delta_{ij}\partial_0^2\Psi} - \cancel{\Box}\,\nabla^2 s_{ij} = \cancel{8\pi G T_{ij}} \end{aligned}}$$

$\Rightarrow \quad w_j = 0 \quad , \quad \Phi = \Psi \quad , \quad s_{ij} = 0 \qquad \left[\text{Using } \begin{cases} \nabla^2 f = 0 \\ f \xrightarrow[|x|\to\infty]{} 0 \end{cases} \Rightarrow f = 0 \right]$

(trace *)

$$\Rightarrow \quad \boxed{ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\,\delta_{ij}\,dx^i dx^j} \quad , \quad \nabla^2\Phi = 4\pi G\rho$$

Photon trajectories

Null geodesic in Minkowski

Pert. $\sim O(\Phi)$

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$$

$$\Big\downarrow \frac{d}{d\lambda}$$

$$p^\mu = \frac{dx^\mu}{d\lambda} = k^\mu + \ell^\mu(\lambda) \,, \qquad k^\mu k_\mu = 0$$

$$(k^0)^2 = (\vec{k})^2 \equiv k^2$$

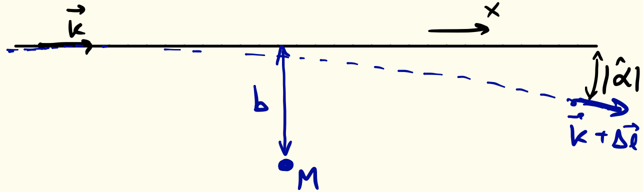
Exercise: show that the geodesic eq. $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$ leads to

$$\begin{cases} \dfrac{d\ell^0}{d\lambda} = -2k\, \vec{k}\cdot\vec{\nabla}\Phi \\[2ex] \dfrac{d\vec{\ell}}{d\lambda} = -2k^2\, \vec{\nabla}_\perp \Phi \end{cases}$$

$$\vec{\nabla}_\perp \equiv \vec{\nabla} - \frac{\vec{k}}{k^2}\,\vec{k}\cdot\vec{\nabla}$$

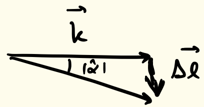
$$\Delta \vec{\ell} = \int d\lambda \, \frac{d\vec{\ell}}{d\lambda} = -2k^2 \int \vec{\nabla}_\perp \Phi \, d\lambda = -2k \int \vec{\nabla}_\perp \Phi \, ds$$

spatial distance
$s = k\lambda$



deflection angle

$$\hat{\alpha} = -\frac{\Delta \vec{\ell}}{k} = 2 \int \vec{\nabla}_\perp \Phi \, ds$$



Exercise: Compute $\hat{\alpha}$ for a star of mass M and impact parameter b

$$\Phi = -\frac{GM}{r} = -\frac{GM}{(b^2+x^2)^{1/2}} \rightarrow \vec{\nabla}_\perp \Phi = \frac{GM\vec{b}}{(b^2+x^2)^{3/2}}$$

$$\hat{\alpha} = 2GM\vec{b} \int \frac{dx}{(b^2+x^2)^{3/2}} = \frac{4GM}{b} \frac{\vec{b}}{b}$$

Observed by Eddington 1919. $\longrightarrow$ Gravitational lensing

Gravitational waves in vacuum

$T_{\mu\nu} = 0$

$$\boxed{\begin{aligned}
&G_{00} = 2\nabla^2\Psi = \cancel{8\pi G T_{00}} \qquad \Rightarrow \quad \Psi = 0 \\
&G_{0j} = -\tfrac{1}{2}\nabla^2 w_j + \cancel{2\partial_0\partial_j\Psi} = \cancel{8\pi G T_{0j}} \qquad \Rightarrow \quad w_j = 0 \\
\circledast \quad &G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \cancel{\Psi}) - \cancel{\partial_0 \partial_{(i} w_{j)}} + \cancel{2\delta_{ij}\partial_0^2\Psi} - \Box S_{ij} = \cancel{8\pi G T_{ij}}
\end{aligned}}$$

$$\text{Trace } \circledast \Rightarrow \nabla^2\Phi = 0 \Rightarrow \Phi = 0$$

$$\Rightarrow \quad \boxed{\Box S_{ij} = 0} \qquad \partial^i S_{ij} = 0$$

Transverse traceless gauge $\Leftrightarrow \quad h^{TT}_{0\nu} = 0 \,, \quad \eta^{\mu\nu} h^{TT}_{\mu\nu} = 0 \,, \quad \partial^\mu h^{TT}_{\mu\nu} = 0$

$$\Box h^{TT}_{\mu\nu} = 0$$

Gravitational waves in vacuum

Transverse Traceless gauge $\Leftrightarrow$ $h_{0\nu}^{TT}=0$, $\eta^{\mu\nu}h_{\mu\nu}^{TT}=0$, $\partial^{\mu}h_{\mu\nu}^{TT}=0$

$$\Box h_{\mu\nu}^{TT}=0 \quad \xrightarrow[\text{plane waves}]{} \quad h_{\mu\nu}^{TT}=C_{\mu\nu}\,e^{ik_\sigma x^\sigma} \quad \text{with} \quad k_\sigma k^\sigma=0$$

$k^\sigma=(\omega,\vec{k})$ frequency, wave-vector

$\omega=|\vec{k}|$

TT gauge $\Rightarrow$ $C_{0\mu}=0$, $\eta^{\mu\nu}C_{\mu\nu}=0$, $k^\mu C_{\mu\nu}=0$

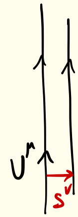
Consider $k^\mu=(\omega,0,0,\omega)$ then $C_{\mu 3}=0$ $\qquad \omega^2=\vec{k}^2+m^2$

$$\Rightarrow \quad C_{\mu\nu}=\begin{pmatrix}0&0&0&0\\0&h_+&h_\times&0\\0&h_\times&-h_+&0\\0&0&0&0\end{pmatrix}\longrightarrow 2 \text{ d.o.f } \checkmark$$

Physical effect of Gravitational waves

Consider the effect on nearby geodesics:

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma$$



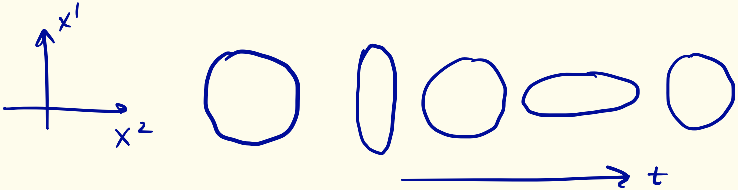
To leading order this gives [using $U^\mu = (1,0,0,0)$]

$$\frac{D^2}{d\tau^2} S^\mu = \frac{1}{2} S^\sigma \partial_0^2 h^{TT\,\mu}{}_\sigma$$

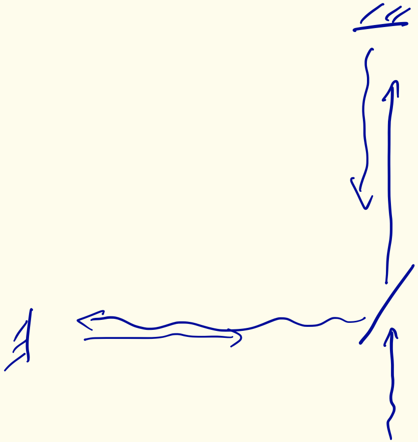
Suppose $h_\times = 0$. Then

$$\frac{D^2}{d\tau^2} s^1 = \frac{1}{2} s^1 \frac{\partial^2}{\partial t^2}\left(h_+ e^{-i\omega t}\right) \Rightarrow S^1(t) = \left(1 + \frac{1}{2} h_+ e^{-i\omega t}\right) s^1(0)$$

$$\frac{D^2}{d\tau^2} s^2 = -\frac{1}{2} s^2 \frac{\partial^2}{\partial t^2}\left(h_+ e^{-i\omega t}\right) \Rightarrow S^2(t) = \left(1 - \frac{1}{2} h_+ e^{-i\omega t}\right) S^2(0)$$



Similarly, $h_+ = 0$ and $h_\times \neq 0$ leads to



Wisdom of the day

Effective Altruism

80000hours.org

Lecture 14

Production of gravitational waves

Recap

Where were we?

General Relativity

Actions

Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}} \longleftarrow \boxed{S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_M}$$

Geodesic equations

$$\boxed{\frac{dx^\nu}{d\tau} \nabla_\nu \frac{dx^\mu}{d\tau} = 0} \longleftarrow \boxed{S = -m \int d\tau}$$

Energy-momentum tensor

Cosmological constant

$$\frac{\Lambda}{8\pi G} = \rho_{vac}$$

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} - \frac{\Lambda}{8\pi G} g_{\mu\nu}}$$

Linearized Gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad , \qquad |h_{\mu\nu}| << 1$$

We shall neglect $O(h^2)$ in the e.o.m.

The Riemann Tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma} \right) \qquad \circledast$$

is invariant under infinitesimal gauge transformations : $[x^\mu \to x^\mu - \xi^\mu]$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Gravitational waves in vacuum

Transverse Traceless gauge $\Longleftrightarrow$ $h_{0\nu}^{TT} = 0 \;,\; \eta^{\mu\nu} h_{\mu\nu}^{TT} = 0 \;,\; \partial^\mu h_{\mu\nu}^{TT} = 0$

$$\boxed{\Box h_{\mu\nu}^{TT} = 0} \xrightarrow[\text{waves}]{\text{plane}} h_{\mu\nu}^{TT} = C_{\mu\nu}\, e^{i k_\sigma x^\sigma} \quad \text{with} \quad k_\sigma k^\sigma = 0$$

$k^\sigma = (\omega, \vec{k})$

frequency

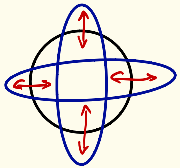
wave-vector

$\omega = |\vec{k}|$

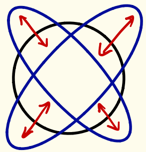
TT gauge $\Rightarrow$ $C_{0\mu} = 0 \;,\; \eta^{\mu\nu} C_{\mu\nu} = 0 \;,\; k^\mu C_{\mu\nu} = 0$

Consider $k^\mu = (\omega, 0, 0, \omega)$ then $C_{\mu 3} = 0$

$$\Rightarrow \quad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow 2 \text{ d.o.f } \checkmark$$



h_+



$h_\times$

Plan for today

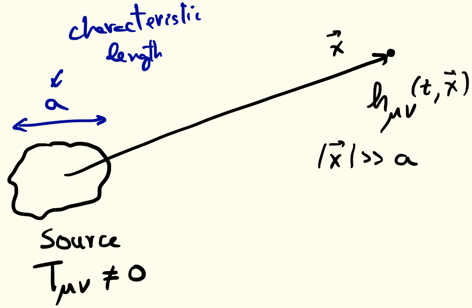
- Radiation gravitational field
- Detection of gravitational waves
- Energy loss due to gravitational radiation
- Laws of Black Hole Mechanics (cont. of lecture 12)

Radiation gravitational field

In analogy with electrodynamics, we are going to compute the gravitational field $h_{\mu\nu}$ far away from the source.

It is convenient to define

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \, , \qquad h = \eta^{\mu\nu} h_{\mu\nu}$$



and impose Lorenz gauge

$$\boxed{\partial_\mu \bar{h}^{\mu\nu} = 0}$$

Exercise: Using ⊛ show that, in this gauge, the E.E. give:

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G \, T_{\mu\nu}}$$

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

This equation can be solved using the retarded Green function:

$$\bar{h}_{\mu\nu}(x) = -16\pi G \int G_R(x-y)\, T_{\mu\nu}(y)\, d^4y \quad , \quad \Box_x G_R(x,y) = \delta^{(4)}(x-y)$$

Recalling that

$$G_R(x-y) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\, \delta\left(|\vec{x}-\vec{y}| - (x^0-y^0)\right) \Theta(x^0-y^0)$$

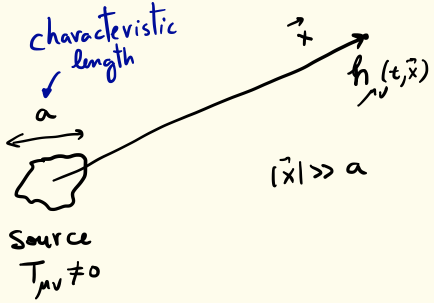
we can write

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G \int \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(\overbrace{t-|\vec{x}-\vec{y}|}^{t_r = \text{retarded time}}, \vec{y})\, d^3y$$

Slow source approximation

$$\frac{\dot{T}_{\mu\nu}}{T_{\mu\nu}} \sim \underbrace{\frac{1}{T}}_{\text{characteristic time}} \ll \frac{c}{a} \iff v \ll c \iff \lambda \gg a$$

$$\bullet \equiv \frac{d}{dt}$$



$$|\vec{x}-\vec{y}| = |\vec{x}| - \vec{n}\cdot\vec{y} + O\left(\frac{1}{|\vec{x}|}\right) \quad , \quad \vec{n} \equiv \frac{\vec{x}}{|\vec{x}|}$$

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G \int \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}\big(\overbrace{t-|\vec{x}-\vec{y}|}^{t_r \,=\, \text{retarded time}}, \vec{y}\big)\, d^3y$$

$$= \frac{4G}{|\vec{x}|} \int d^3y \left[T_{\mu\nu}(t-|\vec{x}|,\vec{y}) - \vec{n}\cdot\vec{y}\, \dot{T}_{\mu\nu}(t-|\vec{x}|,\vec{y}) + \dots \right] + O\left(\frac{1}{|\vec{x}|^2}\right)$$

$$= \frac{4G}{|\vec{x}|} \int d^3y \; T_{\mu\nu}(t-|\vec{x}|,\vec{y}) \left[1 + O\left(\frac{a}{cT}\right) \right] + O\left(\frac{1}{|\vec{x}|^2}\right)$$

$\frac{a}{cT} \ll 1$ slow source

Let us define the quadrupole moment tensor :

$$I_{ij}(t) \equiv \int d^3y \; y_i y_j \; T^{00}(t,\vec{y})$$

Notice that

$$\ddot{I}_{ij} = \frac{d}{dt}\int d^3y \; y_i y_j \underbrace{\partial_0 T^{00}}_{-\partial_k T^{k0}} = \frac{d}{dt}\int d^3y \; T^{0k} \underbrace{\partial_k (y_i y_j)}_{\delta_{ki} y_j + y_i \delta_{kj}} =$$

$$= \frac{d}{dt}\int d^3y \left(T^0{}_i \, y_j + T^0{}_j \, y_i \right) = \int d^3y \left[y_j \underbrace{\partial_0 T^0{}_i}_{-\partial_k T^k{}_i} + (i \leftrightarrow j) \right]$$

$$= \int d^3y \left[T^k{}_i \underbrace{\partial_k y_j}_{\delta_{kj}} + (i \leftrightarrow j) \right] = 2\int d^3y \; T_{ij}$$

Therefore,

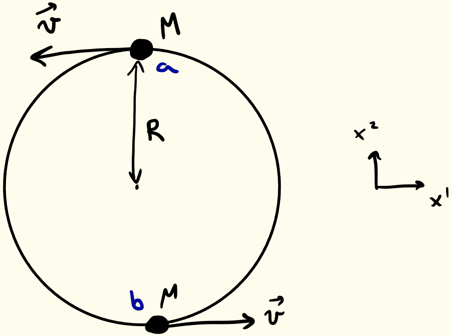
$$\boxed{\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{|\vec{x}|} \ddot{I}_{ij}(t - |\vec{x}|)}$$

Quadrupole formula

$$\boxed{\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{|\vec{x}|}\,\ddot{I}_{ij}(t-|\vec{x}|)}$$

Quadrupole formula

Example : binary system



Newtonian approximation

$$M\frac{v^2}{R} = \frac{GMM}{(2R)^2} \Rightarrow v = \sqrt{\frac{GM}{4R}}$$

$$\Omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi R/v} = \sqrt{\frac{GM}{4R^3}}$$

$$\vec{x}_a = R\left(\cos\Omega t\,,\ \sin\Omega t\right)$$

$$\vec{x}_b = -\vec{x}_a$$

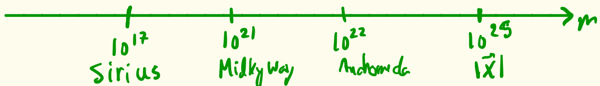
Exercise : show that

$$\bar{h}_{ij}(t,\vec{x}) = \frac{8GM}{|\vec{x}|}\Omega^2R^2\begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_r = t - |\vec{x}|$$

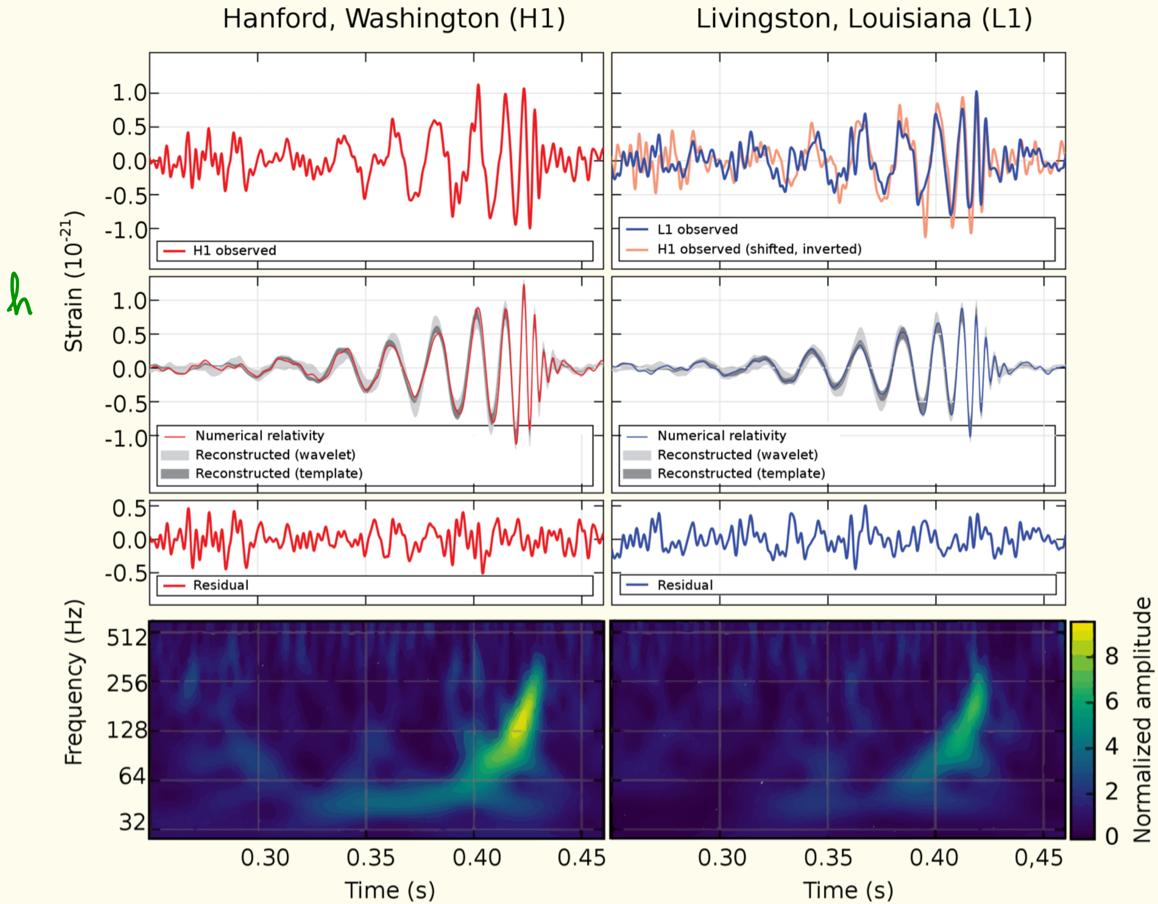
$$\sim \frac{G}{|\vec{x}|}\Omega^2MR^2 \sim \frac{R_s}{|\vec{x}|}v^2 \sim 10^{-21}$$

$M \sim 10\,M_\odot \Rightarrow R_s \sim 10\text{ km} \sim 10^4\text{ m}$

$v \sim c = 1$



First Observation of Gravitational Waves (14/09/2015)

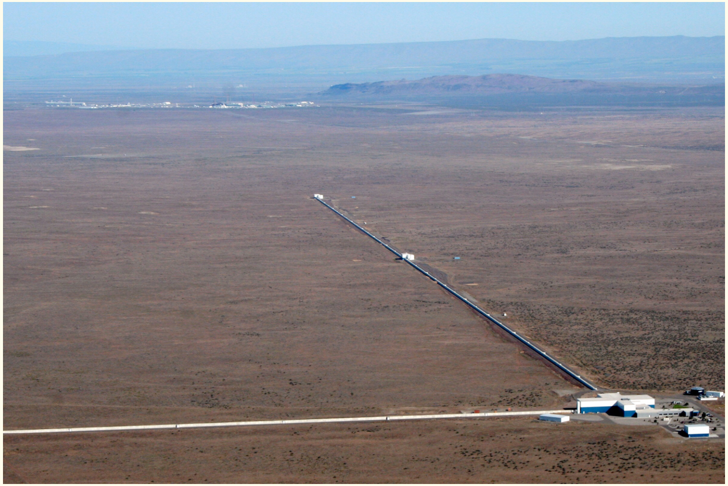


$M_1 \simeq 36\,M_\odot$

$M_2 \simeq 29\,M_\odot$

$\sim 3 M_\odot$ radiated in GWs

LIGO detectors



Hanford

Livingstone

Energy loss due to gravitational radiation

How to associate an energy flux to gravitational waves?

Let's try:

$$G_{\mu\nu}[\overbrace{\eta + h}^{g}] = 8\pi G\, T_{\mu\nu}$$

$$G^{(1)}_{\mu\nu}[h] + \underbrace{G^{(2)}_{\mu\nu}[h]}_{\equiv -8\pi G\, t_{\mu\nu}} + \dots = 8\pi G\, T_{\mu\nu}$$

$$G^{(1)}_{\mu\nu}[h] = 8\pi G\,[T_{\mu\nu} + t_{\mu\nu}]$$

effective energy-momentum tensor (quadratic in h)

It is conserved $\quad \partial_\mu t^{\mu\nu} = 0$

But not gauge invariant $\quad t_{\mu\nu}[h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha] \neq t_{\mu\nu}[h_{\alpha\beta}]$

However, the total energy

$$E = \int d^3x \; t_{00} \qquad \text{is gauge invariant}$$

In addition,

$h = \eta^{\mu\nu} h_{\mu\nu}$

$$t_{\mu\nu} = \frac{1}{32\pi G}\left[\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} - \frac{1}{2}\partial_\mu h\,\partial_\nu h - 2\partial_\rho h^{\rho\sigma}\partial_{(\mu} h_{\nu)\sigma}\right] + \cancel{\text{total derivatives}}$$

Exercise: show that, after dropping the total derivatives, $t_{\mu\nu}$ is gauge invariant.

Energy flux carried by a plane wave

Using transverse traceless (TT) gauge :

$$t_{\mu\nu} = \frac{1}{32\pi G} \partial_\mu h^{TT}_{\rho\sigma} \partial_\nu (h^{TT})^{\rho\sigma}$$

$$h^{TT}_{\mu\nu} = C_{\mu\nu} \cos(k_\sigma x^\sigma)$$ [Taking the real part]

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi G} \underbrace{C_{\rho\sigma} C^{\rho\sigma}}_{2(h_+^2 + h_\times^2)} k_\mu k_\nu \overbrace{\langle \sin^2(k_\sigma x^\sigma) \rangle}^{``1/2"}$$

$$k^\mu = (\omega, 0, 0, \omega)$$

$$= \underbrace{\frac{\omega^2 (h_+^2 + h_\times^2)}{32\pi G}}_{\left(\frac{f}{1\,\mathrm{Hz}}\right)^2 \frac{h_+^2 + h_\times^2}{(10^{-21})^2} \underbrace{10^{-7}\, W/m^2}} \frac{k_\mu k_\nu}{\omega^2}$$

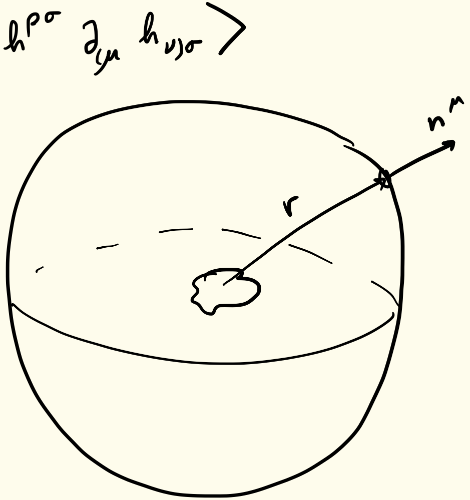
$$\left[\text{frequency } f \equiv \frac{\omega}{2\pi}\right]$$

similar to the energy flux from brightest star on the night sky

Total power radiated

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi G} \left\langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} - \frac{1}{2} \partial_\mu h \partial_\nu h - 2 \partial_\rho h^{\rho\sigma} \partial_{(\mu} h_{\nu)\sigma} \right\rangle$$

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{|\vec{x}|} \ddot{I}_{ij}(t - |\vec{x}|)$$



$$P = \lim_{r \to \infty} \int_{S^2} d\Omega \, r^2 \langle t_{0\mu} \rangle n^\mu$$

After a lengthy calculation, we find

$$\boxed{P = -\frac{G}{5} \left\langle \frac{d^3}{dt^3} J_{ij} \frac{d^3}{dt^3} J^{ij} \right\rangle}$$

energy loss

$$\boxed{J_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} \delta^{kl} I_{kl}}$$

reduced quadrupole moment

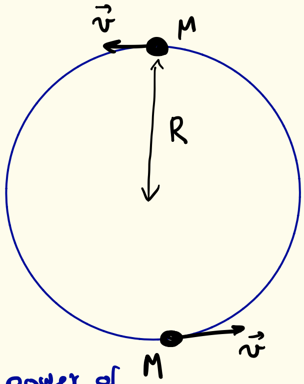
$$J_{ij} \delta^{ij} = 0$$

Total power radiated

$$\boxed{P = -\frac{G}{5}\left\langle \frac{d^3}{dt^3}J_{ij}\,\frac{d^3}{dt^3}J^{ij}\right\rangle}$$

$$\boxed{J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\,\delta^{kl}I_{kl}}$$

Binary system:



$$J_{ij} \sim M R^2$$

$$\frac{d}{dt} \sim \Omega$$

Recall that $\Omega \sim \sqrt{\frac{GM}{R^3}}$

$$P \sim G\,\Omega^6 M^2 R^4$$

$$\sim \frac{G^4 M^5}{R^5} \sim \frac{1}{G}\left(\frac{R_s}{R}\right)^5 \underset{4}{\sim} \frac{1}{G} \sim 10^{52}\,\text{W} > \text{power of all stars in the observable Universe}$$

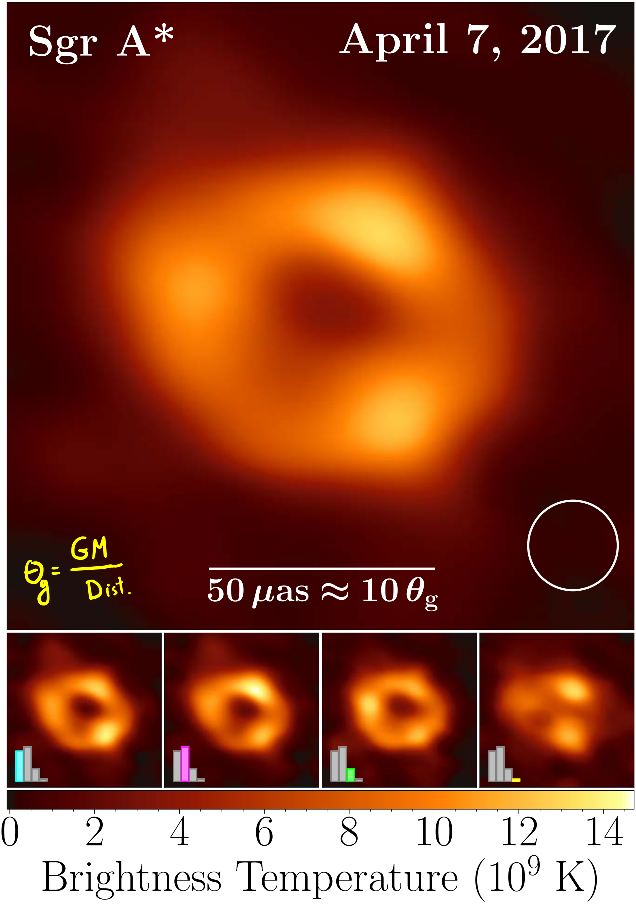
$R_s = 2GM$

$\underset{4}{\sim}$ for BH merging

observed 2015
$P \sim 3\times10^{49}\,\text{W}$

Human dancing:

$$P \sim G\,\underset{1\,\text{Hz}}{\Omega^6}\;\underset{100\,\text{kg}}{M^2}\;\underset{1\,\text{m}}{R^4} \sim 10^{-49}\,\text{W}$$

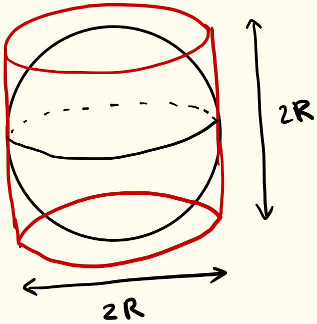


Sgr A*
April 7, 2017
$\theta_g = \frac{GM}{\text{Dist.}}$
$50\,\mu$as $\approx 10\,\theta_{\rm g}$
0
2
4
6
8
10
12
14
Brightness Temperature (10^9 K)

Wisdom of the day

Awe

Archimedes tomb



$$\frac{V_{\ominus}}{V_{\square}} = \frac{\frac{4}{3}\pi R^3}{\pi R^2 \times 2R} = \frac{2}{3}$$

Without calculus !

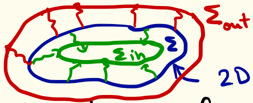
$$\pi = \frac{\text{area } \bigcirc_{R}}{\text{area } \square_{R}}$$

Black Holes

(bonus material)

Do black holes actually form through gravitational collapse?

Probably yes because:



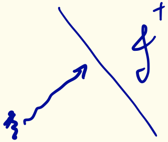
area$(\Sigma) >$ area(Σ_{out})

2D spatial surface

- Singularity Theorems: trapped surface + strong energy condition $\Rightarrow$ singularity

  $\left[T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} T_\lambda^{\ \lambda} t^\sigma t_\sigma \quad \forall t^\mu \text{ timelike} \right]$

  incomplete timelike geodesic



- Cosmic censorship conjecture:

  Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition.

Hawking's area theorem :

- Weak energy condition
  $[T_{\mu\nu} t^\mu t^\nu \geq 0]$
- Cosmic censorship
- asymptotic flatness

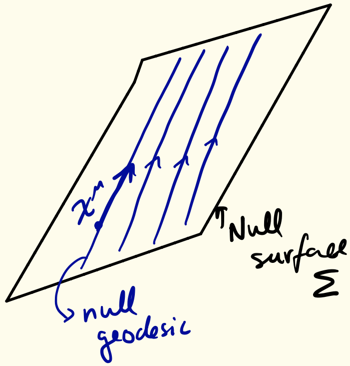
$\Longrightarrow$ Area of future event horizon is non - decreasing

$$\boxed{dA \geq 0}$$

Killing Horizons

Every event horizon in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector field χ^μ.

↳ surface Σ where χ^μ is null.



$$\Rightarrow \quad \chi^\mu \nabla_\mu \chi^\nu = -\kappa \, \chi^\nu \quad (\text{on } \Sigma)$$

κ: surface gravity

$$\chi_\mu \chi^\mu \xrightarrow[r\to\infty]{} -1$$

For Schwarzschild : $\chi^\mu = (\partial_t)^\mu$

Exercise : Show that $\boxed{\kappa = \frac{1}{4GM}}$ for Schwarzschild.

Laws of Black Hole Mechanics

0^{th} law : The surface gravity is constant over the horizon.

1^{st} law : $dM = \frac{\kappa}{8\pi G} dA$

[Check using : $\kappa = \frac{1}{4GM}$, $A = 4\pi(2GM)^2$]

2^{nd} law : $dA \geq 0$

QFT $\Rightarrow$ BH radiates at the Hawking temperature $\boxed{T = \frac{\kappa}{2\pi}}$

$\Rightarrow$ $dM = \frac{\kappa}{8\pi G} dA$ $\Leftrightarrow$ 1^{st} law of thermodynamics

$$\begin{aligned} dM &= \frac{\kappa}{8\pi G} dA \\ \parallel & \\ dE &= T\, d\left(\frac{A}{4G}\right) \end{aligned}$$

$\Rightarrow$ BH entropy : $S_{BH} = \frac{\text{Area}}{4G} = \log(\#\text{ of states})$

what are these?

Holography ...